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THE ROLLING OF A RIGID BODY ON A MOVING SURFACE†

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The problem of the motion of a material system, consisting of a supporting rigid body, bounded by a surface and rolling on another moving surface, and a set of supported point masses, the position of which with respect to this body can be specified by a finite number of generalized coordinates, is considered using methods described previously in [1–3].

1. THE KINEMATIC PROPERTIES OF THE MOTION

We begin our study of the material system considered by describing the notation and by considering the kinematic properties. We introduce a system of rectangular coordinates $Oxyz(\mathbf{i}_1, \mathbf{i}_2$, and $\mathbf{i}_3$ are the unit vectors of the axes) and $O_cx^cy^cz^c$ ($\mathbf{i}_1^c$, $\mathbf{i}_2^c$ and $\mathbf{i}_3^c$ are the unit vectors of the axes), permanently connected with the supporting rigid body and with the surface-base respectively (all systems of coordinates are left systems). This enables us to define the position of the supporting body with respect to the surface-base S^c by the coordinates x_0^c, y_0^c, z_0^c of the point O in the axes $O_cx^cy^cz^c$ and the Euler angles φ, ψ, θ (pure rotation, precession and nutation) between the axes introduced, and the position of the system of supported point masses M_i with respect to the supporting body (with respect to the axes $Oxyz$) – by certain generalized coordinates $\alpha^1, \ldots, \alpha^n$. We denote the projections of the vector of the velocity V_0 of the point O and the vector of the angular velocity of the supporting body ω onto the $Oxyz$ axes by k, l, m and p, q, r. We stipulate that the subscript of the radius vector, having its origin at the point O_a, O_c, O and C_r, and its projections onto the axes $O_ax^ay^az^a$, $O_cx^cy^cz^c$, $Oxyz$ and $C_rx^ry^rz^r$ respectively, is the symbol of the end of the radius vector, while the superscript is the symbol of its origin O_a, O_c, O and C_r (the symbol of the system of coordinates $O_ax^ay^az^a$, $O_cx^cy^cz^c$, $Oxyz$ and $C_rx^ry^rz^r$).

We will assume further that r_i^c and r_0^c are the radius vectors, which fix the position of the points M_i and O on the axes $O_cx^cy^cz^c$, while r_i and r_C are the radius vectors which fix the position of the points M_i and C (C is the centre of inertia of the system) on the axes $Oxyz$; hence we obtain

$$\mathbf{r}_i^c = \mathbf{r}_0^c(x_0^cy_0^cz_0^c) + \mathbf{r}_i(\alpha^1, \ldots, \alpha^n)$$

We will now assume‡ that the surface-base S^c moves, and its motion (the motion of the system of coordinates $O_cx^cy^cz^c$ permanently connected with it) with respect to the fixed system of coordinates $O_ax^ay^az^a$ ($\mathbf{i}_1^a$, $\mathbf{i}_2^a$ and $\mathbf{i}_3^a$ are the orthonormalized vectors of the axes) is known, i.e. the radius vector $\mathbf{r}_{O_c}^a = \mathbf{i}_1^a x_{O_c}^a + \mathbf{i}_2^a y_{O_c}^a + \mathbf{i}_3^a z_{O_c}^a$ of the point O_c with origin at the point O_a and the parameters (the generalized coordinates), defining the orientation of the axes $O_cx^cy^cz^c$ with respect to the axes $O_ax^ay^az^a$ (for example, the Euler angles φ_c, ψ_c, θ_c) are specified as functions of time.

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‡A more complete discussion can be found in the following preprints by the author: Rolling of a rigid body on a moving surface. Russian Academy of Sciences, Moscow, 1995; To the problem of the rolling of a rigid body on a moving surface. Institute of Mechanics, Moscow State University, Moscow, 1998; Rolling of a sphere on a moving plane. Moscow State University, Moscow, 2001.

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Further, introducing the radius vector ρ with the origin at the point O and the Gaussian coordinates q^1, q^2 for points of the surface S, bounding the supporting body, we will specify its equation in the form

$$\boldsymbol{\rho} = \boldsymbol{\rho}(q^1, q^2) \ (\boldsymbol{\rho} = x\mathbf{i}_1 + y\mathbf{i}_2 + z\mathbf{i}_3),$$

while the coefficients of the first two quadratic forms will be denoted by a_{11}, a_{22}, b_{11}, and b_{22} (for simplicity we will assume that the coordinate lines of the surface are lines of curvature). At the contact point M we will attach to the surface S the moving frame of reference Mq^1q^2n with the unit vectors directed along the tangent to the coordinate lines and the normal

$$\mathbf{e}_1 = \frac{1}{\sqrt{a_{11}}}\boldsymbol{\rho}_1, \quad \mathbf{e}_2 = \frac{1}{\sqrt{a_{22}}}\boldsymbol{\rho}_2, \quad \mathbf{e}_3 = \frac{1}{\sqrt{a_{11}a_{22}}}(\boldsymbol{\rho}_1 \times \boldsymbol{\rho}_2) \quad \left(\boldsymbol{\rho}_\alpha = \frac{\partial}{\partial q^\alpha}\boldsymbol{\rho}\right)$$

We will denote the projections of the vectors $\mathbf{r}_C$ and $\boldsymbol{\rho}$ onto the axis of this frame of reference by

$$\xi_C, \eta_C, \varepsilon_C, \xi = \frac{1}{\sqrt{a_{11}}}\rho\frac{\partial\rho}{\partial q^1}, \quad \eta = \frac{1}{\sqrt{a_{22}}}\rho\frac{\partial\rho}{\partial q^2}, \quad \varepsilon(\rho^2 = x^2 + y^2 + z^2)$$

We will introduce the cosines of the angles between the axes $O_ax^ay^az^a$ and $O_cx^cy^cz^c$, between the axes $O_cx^cy^cz^c$ and $Mq_c^1q_c^2n$, and also between the axes Mq^1q^2n and $Oxyz$

$$\begin{aligned}
\mathbf{i}_1^c &= l_a^1\mathbf{i}_1^a + m_a^1\mathbf{i}_2^a + n_a^1\mathbf{i}_3^a, & \mathbf{e}_1^c &= \alpha_c\mathbf{i}_1^c + \alpha_c'\mathbf{i}_2^c + \alpha_c''\mathbf{i}_3^c \\
\mathbf{i}_2^c &= l_a^2\mathbf{i}_1^a + m_a^2\mathbf{i}_2^a + n_a^2\mathbf{i}_3^a, & \mathbf{e}_2^c &= \beta_c\mathbf{i}_1^c + \beta_c'\mathbf{i}_2^c + \beta_c''\mathbf{i}_3^c \\
\mathbf{i}_3^c &= l_a^3\mathbf{i}_1^a + m_a^3\mathbf{i}_2^a + n_a^3\mathbf{i}_3^a, & \mathbf{e}_3^c &= \gamma_c\mathbf{i}_1^c + \gamma_c'\mathbf{i}_2^c + \gamma_c''\mathbf{i}_3^c \\
\mathbf{i}_1 &= \alpha\mathbf{e}_1 + \beta\mathbf{e}_2 + \gamma\mathbf{e}_3, \quad \mathbf{i}_2 = \alpha'\mathbf{e}_1 + \beta'\mathbf{e}_2 + \gamma'\mathbf{e}_3, & \mathbf{i}_3 &= \alpha''\mathbf{e}_1 + \beta''\mathbf{e}_2 + \gamma''\mathbf{e}_3
\end{aligned} \tag{1.1}$$

All that has been stated here for the surface S, which bounds the supporting body, also holds for the surface-base S^c (the corresponding values are denoted by the same letters but with an index c). Further, following Voronets, we will define the position of the supporting body by generalized coordinates q^1, q^2, q_c^1, q_c^2 and ϑ (the first four quantities are the Gaussian coordinates of the point M, and ϑ is the angle between the axes q^1 and q_c^2 at the same point), while the position of the whole system, consequently, will be defined by the generalized coordinates $q^1, q^2, q_c^1, q_c^2, \vartheta, \alpha^1, \ldots, \alpha^n$.

The projections K_c, l_c and m_c of the velocity $\mathbf{v}_{Oc}$ of the point O_c onto the $O_cx^cy^cz^c$ axes will be as follows:

$$k_c \equiv \mathbf{i}_1^c \cdot \mathbf{v}_{O_c} = l_a^1\dot{x}_{O_c}^a + m_a^1\dot{y}_{O_c}^a + n_a^1\dot{z}_{O_c}^a \tag{1.2}$$

where l_c and m_c are obtained from relations (1.2) by replacing the superscript 1 by 2 and 4, while the projections p_c, q_c and r_c of the angular velocity vector $\boldsymbol{\omega}_c$ of the surface-base onto the axes $O_cx^cy^cz^c$ are given by known formulae [3, formulae (2.9.3)], where we must replace φ, ψ, θ by $\varphi_c, \psi_c, \theta_c$).

For the projections of the velocity $\mathbf{j} = \mathbf{v}_{O_c} + \boldsymbol{\omega}_c \times \boldsymbol{\rho}^c$ of a point the surface-base S^c, coinciding at the given instant with the contact point M, onto the axes $\mathbf{i}_1^c, \mathbf{i}_2^c, \mathbf{i}_3^c$ and $\mathbf{e}_1^c, \mathbf{e}_2^c, \mathbf{e}_3^c$, we correspondingly obtain

$$\begin{aligned}
f_1 &= k_c + q_cz^c - r_cy^c, & b_1 &= \mathbf{e}_1^c \cdot \mathbf{j} = \alpha_cf_1 + a_c'f_2 + \alpha_c''f_3 \\
f_2 &= l_c + r_cx^c - p_cz^c, & b_2 &= \mathbf{e}_2^c \cdot \mathbf{j} = \beta_cf_1 + \beta_c'f_2 + \beta_c''f_3 \\
f_3 &= m_c + p_cy^c - q_cx^c, & b_3 &= \mathbf{e}_3^c \cdot \mathbf{j} = \gamma_cf_1 + \gamma_c'f_2 + \gamma_c''f_3
\end{aligned} \tag{1.3}$$

where $f_1, f_2, f_3, b_1, b_2, b_3$ are functions of t, q_c^1, q_c^2. Finally, we obtain as functions of $t, q_c^1, q_c^2, \vartheta$ the projections j_1, j_2, j_3 of the vector $\mathbf{j}$ onto the axes of the moving frame of reference Mq^1q^2n (j_k occurs in the equation of motion)

$$j_1 = \pm b_1\sin\vartheta + b_2\cos\vartheta, \quad j_2 = \mp b_1\cos\vartheta + b_2\sin\vartheta, \quad j_3 = \pm b_3$$

Here we have used the formulae

$$\mathbf{e}_1 = \pm\mathbf{e}_1^c\sin\vartheta + \mathbf{e}_2^c\cos\vartheta, \quad \mathbf{e}_2 = \mp\mathbf{e}_1^c\cos\vartheta + \mathbf{e}_2^c\sin\vartheta, \quad \mathbf{e}_3 = \pm\mathbf{e}_3^c \tag{1.4}$$

We will put an x above a vector as the symbol of the derivative of the vector in the system of coordinates $O_cx^cy^cz^c$ with respect to time: $\overset{\times c}{\boldsymbol{\rho}}$ (as similarly done by Lur'ye [3], who used an asterisk). The absolute velocity of the contact point (see [3, pp. 81 and 88]) can be written in the form

$$\begin{aligned} \mathbf{v}_M^{\text{abs}} &= \mathbf{v}_{O_c} + \boldsymbol{\omega}_c\times\boldsymbol{\rho}^c + \overset{\times c}{\boldsymbol{\rho}^c}, \quad \overset{\times c}{\boldsymbol{\rho}^c} = \boldsymbol{\rho}_\alpha^c\dot{q}_c^\alpha = \mathbf{e}_1^c\sqrt{a_{11}^c}\dot{q}_c^1 + \mathbf{e}_2^c\sqrt{a_{22}^c}\dot{q}_c^2 \\ \mathbf{v}_M^{\text{abs}} &= \mathbf{v}_0 + \boldsymbol{\omega}\times\boldsymbol{\rho} + \overset{*}{\boldsymbol{\rho}}, \quad \overset{*}{\boldsymbol{\rho}} = \boldsymbol{\rho}_\alpha\dot{q}^\alpha = \mathbf{e}_1\sqrt{a_{11}}\dot{q}^1 + \mathbf{e}_2\sqrt{a_{22}}\dot{q}^2 \end{aligned} \tag{1.5}$$

This gives

$$\mathbf{v}_{O_c} + \boldsymbol{\omega}_c\times\boldsymbol{\rho}^c + \overset{\times c}{\boldsymbol{\rho}^c} = \mathbf{v}_0 + \boldsymbol{\omega}\times\boldsymbol{\rho} + \overset{*}{\boldsymbol{\rho}}$$

We will introduce a vector $\mathbf{U}$ in the plane of the axes $\mathbf{e}_1$, $\mathbf{e}_2$

$$\overset{\times c}{\boldsymbol{\rho}^c} - \overset{*}{\boldsymbol{\rho}} = \mathbf{v}_0 + \boldsymbol{\omega}\times\boldsymbol{\rho} - \mathbf{v}_{O_c} - \boldsymbol{\omega}_c\times\boldsymbol{\rho}^c = \bar{\mathbf{U}} \tag{1.6}$$

Hence ($\boldsymbol{\Phi} \equiv \mathbf{U} + \mathbf{j}$)

$$\mathbf{v}_0 = [\mathbf{U} + (\mathbf{v}_{O_c} + \boldsymbol{\omega}_c\times\boldsymbol{\rho}^c)] - \boldsymbol{\omega}\times\boldsymbol{\rho} = \boldsymbol{\Phi} - \boldsymbol{\omega}\times\boldsymbol{\rho} \tag{1.7}$$

There is no slipping at the point M ($\mathbf{v}_0 + \boldsymbol{\omega}\times\boldsymbol{\rho} - \mathbf{v}_{O_c} - \boldsymbol{\omega}_c\times\boldsymbol{\rho}_c = 0$), and hence we obtain equations of non-holonomic constraint (we project the vector $\mathbf{U} = \overset{\times c}{\boldsymbol{\rho}^c} - \overset{*}{\boldsymbol{\rho}} = 0$ onto the axes $\mathbf{e}_1$, and $\mathbf{e}_2$)

$$\begin{aligned} U^1 &= \pm\sqrt{a_{11}^c}\dot{q}_c^1\sin\vartheta + \sqrt{a_{22}^c}\dot{q}_c^2\cos\vartheta - \sqrt{a_{11}}\dot{q}^1 = 0 \\ U^2 &= \mp\sqrt{a_{11}^c}\dot{q}_c^1\cos\vartheta + \sqrt{a_{22}^c}\dot{q}_c^2\sin\vartheta - \sqrt{a_{22}}\dot{q}^2 = 0 \end{aligned} \tag{1.8}$$

We can represent the angular velocity $\boldsymbol{\omega}$ of the supporting rigid body and the vector of an infinitesimal rotation $\boldsymbol{\theta}$ of this body as [1, p. 804]

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{\omega}_3 + \boldsymbol{\omega}_c, \quad \boldsymbol{\theta} = \boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 + \boldsymbol{\theta}_3 + \boldsymbol{\theta}_c \tag{1.9}$$

The vector $\boldsymbol{\omega}_c$ has projections p_c, q_c, r_c (as also φ_c, ψ_c, θ_c), which are specified functions of time, and hence, at a fixed instant of time, the vector of an infinitesimal rotation $\boldsymbol{\theta}_c = 0$.

Hence, from formulae (1.7) and (1.8) of [1] we obtain expressions for the projections of the angular velocity of the supporting body onto the $\mathbf{e}_i$ axis of the mobile frame of reference (here and henceforth the upper (lower) sign denotes the case $\mathbf{e}_3 = \mathbf{e}_3^c$ ($\mathbf{e}_3 = -\mathbf{e}_3^c$))

$$\begin{aligned} U^3 \equiv \sigma &= -\frac{b_{22}}{\sqrt{a_{22}}}\dot{q}^2 \pm \frac{b_{22}^c}{\sqrt{a_{22}^c}}\dot{q}_c^2\sin\vartheta - \frac{b_{11}^c}{\sqrt{a_{11}^c}}\dot{q}_c^1\cos\vartheta \pm \\ &\pm (p_c\alpha_c + q_c\alpha_c' + r_c\alpha_c'')\sin\vartheta + (p_c\beta_c + q_c\beta_c' + r_c\beta_c'')\cos\vartheta \\ U^4 \equiv \tau &= \frac{b_{11}}{\sqrt{a_{11}}}\dot{q}^1 - \frac{b_{11}^c}{\sqrt{a_{11}^c}}\dot{q}_c^1\sin\vartheta \mp \frac{b_{22}^c}{\sqrt{a_{22}^c}}\dot{q}_c^2\cos\vartheta + \\ &+ (p_c\beta_c + q_c\beta_c' + r_c\beta_c'')\sin\vartheta \mp (p_c\alpha_c + q_c\alpha_c' + r_c\alpha_c'')\cos\vartheta \\ U^5 = n &= \frac{1}{2\sqrt{a_{11}a_{22}}}\left(\frac{\partial a_{11}}{\partial q^2}\dot{q}^1 - \frac{\partial a_{22}}{\partial q^1}\dot{q}^2\right) \mp \frac{1}{2\sqrt{a_{11}^c a_{22}^c}}\left(\frac{\partial a_{11}^c}{\partial q_c^2}\dot{q}_c^1 - \frac{\partial a_{22}^c}{\partial q_c^1}\dot{q}_c^2\right) - \\ &- \dot{\vartheta} \pm (p_c\gamma_c + q_c\gamma_c' + r_c\gamma_c'') \end{aligned} \tag{1.10}$$

We similarly obtain the projections of the vector $\boldsymbol{\theta} = \delta V^3\bar{e}_1 + \delta V^4\bar{e}_2 + \delta V^5\bar{e}_3$

$$\delta V^3 = -\frac{b_{22}}{\sqrt{a_{22}}}\delta q^2 \pm \frac{b_{22}^c}{\sqrt{a_{22}^c}}\delta q_c^2 \sin\vartheta - \frac{b_{11}^c}{\sqrt{a_{11}^c}}\delta q_c^1 \cos\vartheta$$

$$\delta V^4 = \frac{b_{11}}{\sqrt{a_{11}}}\delta q^1 - \frac{b_{11}^c}{\sqrt{a_{11}^c}}\delta q_c^1 \sin\vartheta \mp \frac{b_{22}^c}{\sqrt{a_{22}^c}}\delta q_c^2 \cos\vartheta$$

$$\delta V^5 = \frac{1}{2\sqrt{a_{11}a_{22}}}\left(\frac{\partial a_{11}}{\partial q^2}\delta q^1 - \frac{\partial a_{22}}{\partial q^1}\delta q^2\right) \mp \frac{1}{2\sqrt{a_{11}^c a_{22}^c}}\left(\frac{\partial a_{11}^c}{\partial q_c^2}\delta q_c^1 - \frac{\partial a_{22}^c}{\partial q_c^1}\delta q_c^2\right) - \delta\vartheta$$

Using equations of the constraint (1.8), expressions (1.10) can be converted to the form

$$\begin{aligned} \sigma &= -\Delta_{12}\sqrt{a_{11}}\dot{q}^1 - \Delta_{22}\sqrt{a_{22}}\dot{q}^2 + A, \quad \tau = \Delta_{11}\sqrt{a_{11}}\dot{q}^1 + \Delta_{21}\sqrt{a_{22}}\dot{q}^2 + B \\ n &= -\dot{\vartheta} + \Delta_1\sqrt{a_{11}}\dot{q}^1 - \Delta_2\sqrt{a_{22}}\dot{q}^2 + C \end{aligned} \tag{1.11}$$

where

$$A = \pm\sigma_c \sin\vartheta + \tau_c \cos\vartheta, \quad \sigma_c = p_c\alpha_c + q_c\alpha_c' + r_c\alpha_c''$$

$$B = \mp\sigma_c \cos\vartheta + \tau_c \sin\vartheta, \quad \tau_c = \ldots$$

$$C = \pm n_c, \quad n_c = p_c\gamma_c + q_c\gamma_c' + r_c\gamma_c''$$

$$\Delta_{11} = \frac{b_{11}}{a_{11}} \mp \frac{b_{11}^c}{a_{11}^c}\sin^2\vartheta \mp \frac{b_{22}^c}{a_{22}^c}\cos^2\vartheta, \quad \Delta_{22} = \frac{b_{22}}{a_{22}} \mp \frac{b_{22}^c}{a_{22}^c}\sin^2\vartheta \mp \frac{b_{11}^c}{a_{11}^c}\cos^2\vartheta$$

$$2\Delta_1 = \frac{1}{\sqrt{a_{22}}}\frac{\partial \ln a_{11}}{\partial q^2} - \frac{\sin\vartheta\,\partial \ln a_{11}^c}{\sqrt{a_{22}^c}\,\partial q_c^2} \pm \frac{\cos\vartheta\,\partial \ln a_{22}^c}{\sqrt{a_{11}^c}\,\partial q_c^1}$$

$$2\Delta_2 = \frac{1}{\sqrt{a_{11}}}\frac{\partial \ln a_{22}}{\partial q^1} \mp \frac{\sin\vartheta\,\partial \ln a_{22}^c}{\sqrt{a_{11}^c}\,\partial q_c^1} - \frac{\cos\vartheta\,\partial \ln a_{11}^c}{\sqrt{a_{22}^c}\,\partial q_c^2}$$

$$\Delta_{12} \equiv \Delta_{21} = \mp\left(\frac{b_{22}^c}{a_{22}^c} - \frac{b_{11}^c}{a_{11}^c}\right)\sin\vartheta\cos\vartheta$$

We point out the following formulae

$$p = \sigma\alpha + \tau\beta + n\gamma, \quad q = \sigma\alpha' + \tau\beta' + n\gamma', \quad r = \sigma\alpha'' + \tau\beta'' + n\gamma''$$

From relations (1.8) and (1.10) we further determine the expressions for $\dot{q}^1, \dot{q}^2, \dot{q}_c^1, \dot{q}_c^2$ and $\dot{\vartheta}$ in terms of the quasi-velocity (the terms with U^1 and U^2 are not written out)

$$\begin{aligned} \dot{q}^1 &= \frac{1}{\sqrt{a_{11}}R}(\sigma\Delta_{12} + \tau\Delta_{22} - A\Delta_{12} - B\Delta_{22}), \quad \dot{q}^2 = -\frac{1}{\sqrt{a_{22}}R}(\sigma\Delta_{11} + \tau\Delta_{21} - A\Delta_{11} - B\Delta_{21}) \\ \dot{q}_c^1 &= \frac{1}{\sqrt{a_{11}^c}d}(\sigma c_{12} + \tau c_{22} - Ac_{12} - Bc_{22}), \quad \dot{q}_c^2 = -\frac{1}{\sqrt{a_{22}^c}d}(\sigma c_{11} + \tau c_{21} - Ac_{11} - Bc_{21}) \\ \dot{\vartheta} &= -n + \frac{(\sigma - A)}{R}(\Delta_{12}\Delta_1 + \Delta_{11}\Delta_2) - \frac{(\tau - B)}{R}(\Delta_{22}\Delta_1 + \Delta_{21}\Delta_2) + C \end{aligned} \tag{1.12}$$

Here

$$d = c_{11}c_{22} - c_{12}c_{21} = \pm R, \quad R = \Delta_{11}\Delta_{22} - \Delta_{12}^2$$

$$c_{12} = \left(\frac{b_{11}}{a_{11}} \mp \frac{b_{22}^c}{a_{22}^c}\right)\cos\vartheta, \quad c_{11} = \left(\pm\frac{b_{11}}{a_{11}} - \frac{b_{11}^c}{a_{11}^c}\right)\sin\vartheta$$
$$c_{22} = \left(\frac{b_{22}}{a_{22}} \mp \frac{b_{22}^c}{a_{22}^c}\right)\sin\vartheta, \quad c_{21} = -\left(\pm\frac{b_{22}}{a_{22}} - \frac{b_{11}^c}{a_{11}^c}\right)\cos\vartheta \tag{1.13}$$

Finally, we obtain, from well-known formulae (see [3, p. 160]) and formula (1.6), an expression for the kinetic energy T' of the system considered, derived without taking into account the equations of the constraint

$$2T' = Mv_0^2 + 2M\mathbf{v}_0 \cdot (\boldsymbol{\omega} \times \mathbf{r}_C) + \boldsymbol{\omega} \cdot \boldsymbol{\Theta}^O \cdot \boldsymbol{\omega} + 2(\mathbf{v}_0 \cdot \mathbf{Q}_r + \boldsymbol{\omega} \cdot \mathbf{K}_r^O) + \sum_{i=1} m_i v_i^2 =$$
$$= M\boldsymbol{\Phi}^2 - 2M\boldsymbol{\Phi} \cdot (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \boldsymbol{\omega} \cdot \boldsymbol{\Theta}^O \cdot \boldsymbol{\omega} + M\rho^2\omega^2 - M(\boldsymbol{\rho} \cdot \boldsymbol{\omega})^2 +$$
$$+ 2M\boldsymbol{\Phi} \cdot (\boldsymbol{\omega} \times \mathbf{r}_C) - 2M\mathbf{r}_C \cdot [\boldsymbol{\rho}\omega^2 - \boldsymbol{\omega}(\boldsymbol{\omega} \cdot \boldsymbol{\rho})] +$$
$$+ 2\boldsymbol{\Phi} \cdot \mathbf{Q}_r + 2\boldsymbol{\omega} \cdot (\mathbf{Q}_r \times \boldsymbol{\rho} + \mathbf{K}_r^O) + \sum_{i=1}^N m_i v_i^2 = M[U^2 + 2\mathbf{U} \cdot \mathbf{j} + j^2] - \tag{1.14}$$
$$- 2M[\mathbf{U} \cdot (\boldsymbol{\omega} \times \boldsymbol{\rho}) + \mathbf{j} \cdot (\boldsymbol{\omega} \times \boldsymbol{\rho})] + \boldsymbol{\omega} \cdot \boldsymbol{\Theta}^O \cdot \boldsymbol{\omega} + M\rho^2\omega^2 - M(\boldsymbol{\rho} \cdot \boldsymbol{\omega})^2 +$$
$$+ 2M[\mathbf{U} \cdot (\bar{\boldsymbol{\omega}} \times \bar{\mathbf{r}}_C) + \mathbf{j} \cdot (\boldsymbol{\omega} \times \mathbf{r}_C)] - 2M\mathbf{r}_C \cdot [\boldsymbol{\rho}\omega^2 - \boldsymbol{\omega}(\boldsymbol{\omega} \cdot \boldsymbol{\rho})] +$$
$$+ 2[\mathbf{U} \cdot \mathbf{Q}_r + \mathbf{j} \cdot \mathbf{Q}_r] + 2\boldsymbol{\omega} \cdot (\mathbf{Q}_r \times \boldsymbol{\rho} + \mathbf{K}_r^O) + \sum_{i=1}^N m_i v_i^2 (\mathbf{v}_i = \overset{*}{\mathbf{r}}_i)$$

and an expression for Θ', derived taking into account the equations of the constraint for $\mathbf{U} = 0$.

We will denote the vector with projections $\partial\Theta'/\partial U^3$, $\partial\Theta'/\partial U^4$, $\partial\Theta'/\partial U^5$ onto the axis Mq^1q^2n by $\mathbf{m}'$. The expressions for the kinetic energy of the system $T' = T + T''$, $\Theta' = \Theta + \Theta''$ and the vector $\mathbf{m}' = \mathbf{m} + \mathbf{m}''$ can be split into two terms, where

$$2T'' = 2\mathbf{j}(M\mathbf{U} + \mathbf{Q}_r) + Mj^2 + 2M\mathbf{j} \cdot [(\boldsymbol{\omega} \times \mathbf{r}_C) - (\boldsymbol{\omega} \times \boldsymbol{\rho})]$$
$$2\Theta'' = 2\mathbf{j} \cdot \mathbf{Q}_r + Mj^2 + 2M\mathbf{j} \cdot [(\boldsymbol{\omega} \times \mathbf{r}_C) - (\boldsymbol{\omega} \times \boldsymbol{\rho})] \tag{1.15}$$
$$\mathbf{m}'' = -M(\boldsymbol{\rho} \times \mathbf{j}) + M(\mathbf{r}_C \times \mathbf{j})$$

The expression for $2T$ is given by formula (2.9) in [1], where we must replace $\boldsymbol{\Omega}$ by new notation: $\mathbf{U}$ and

$$\mathbf{m} = \boldsymbol{\Theta}^O \cdot \boldsymbol{\omega} + M\rho^2\boldsymbol{\omega} - M(\boldsymbol{\rho} \cdot \boldsymbol{\omega})\boldsymbol{\rho} - 2M(\mathbf{r}_C \cdot \boldsymbol{\rho})\boldsymbol{\omega} +$$
$$+ M(\boldsymbol{\omega} \cdot \mathbf{r}_C)\boldsymbol{\rho} + M(\boldsymbol{\omega} \cdot \boldsymbol{\rho})\mathbf{r}_C + \mathbf{Q}_r \times \boldsymbol{\rho} + \mathbf{K}_r^O \tag{1.16}$$

Incidentally, if we are given the vectors

$$\boldsymbol{\rho} = x\mathbf{i}_1 + y\mathbf{i}_2 + z\mathbf{i}_3, \quad \mathbf{e}_1 = \alpha\mathbf{i}_1 + \alpha'\mathbf{i}_2 + \alpha''\mathbf{i}_3$$
$$\boldsymbol{\omega} = p\mathbf{i}_1 + q\mathbf{i}_2 + r\mathbf{i}_3, \quad \mathbf{j} = j_x \cdot \mathbf{i}_1 + j_y \cdot \mathbf{i}_2 + j_z \cdot \mathbf{i}_3$$

we have

$$\frac{\partial \mathbf{p}}{\partial q^\alpha} = \frac{\partial x}{\partial q^\alpha}\mathbf{i}_1 + \ldots + \frac{\partial z}{\partial q^\alpha}\mathbf{i}_3, \quad \frac{\partial \mathbf{e}_1}{\partial q^\alpha} = \frac{\partial \alpha}{\partial q^\alpha}\mathbf{i}_1 + \ldots + \frac{\partial \alpha''}{\partial q^\alpha}\mathbf{i}_3$$
$$\frac{\partial \boldsymbol{\omega}}{\partial q^\alpha} = \frac{\partial p}{\partial q^\alpha}\mathbf{i}_1 + \ldots + \frac{\partial r}{\partial q^\alpha}\mathbf{i}_3, \quad \frac{\partial \mathbf{j}}{\partial q^\alpha} = \frac{\partial j_x}{\partial q^\alpha}\mathbf{i}_1 + \ldots + \frac{\partial j_z}{\partial q^\alpha}\mathbf{i}_3. \tag{1.17}$$

2. THE EQUATIONS OF MOTION

Now, taking expressions (1.10) and (1.8) as the quasivelocities

$$U^1 = a^1, \quad U^2 = a^2, \quad U^3 = \sigma, \quad U^4 = \tau, \quad U^5 = n, \quad \dot\alpha^1, \ldots, \dot\alpha^n$$

and denoting the corresponding variations of the quasicoordinates by δV^i, $\delta\alpha^k$ using the Euler–Lagrange equations we can derive the equations of motion of the system, following Lar'ye [3]. Proceeding as in [1] using the notation and calculations used there, we obtain that only the following symbols Γ and ε are non-zero.

$$\Gamma^s_{35} = -\Gamma^s_{53} = \kappa^s_1, \quad \Gamma^s_{45} = -\Gamma^s_{54} = \kappa^s_2; \quad s = 1, 2$$
$$\varepsilon^s_3 = \frac{\Delta_{s1}n_c}{d}, \quad \varepsilon^s_4 = \frac{\Delta_{s2}n_c}{d}, \quad \varepsilon^s_5 = -\kappa^s_1 A - \kappa^s_2 B \tag{2.1}$$

$$\Gamma^3_{35} = -\Gamma^3_{53} = q_1, \quad \Gamma^3_{45} = -\Gamma^3_{54} = -1 + q_2, \quad \Gamma^3_{34} = -\Gamma^3_{43} = -r_1$$
$$\Gamma^4_{35} = -\Gamma^4_{53} = 1 - p_1, \quad \Gamma^4_{45} = -\Gamma^4_{54} = -p_2, \quad \Gamma^4_{34} = -\Gamma^4_{43} = -r_2$$
$$\Gamma^5_{34} = -\Gamma^5_{43} = p_1 + q_2 - 1 \quad \text{or} \quad \Gamma^\varepsilon_{\alpha\beta} = \gamma^{\varepsilon+1}_{\alpha+1,\beta+1} \tag{2.2}$$
$$\varepsilon^3_4 = -r_3, \quad \varepsilon^3_5 = q_3, \quad \varepsilon^4_3 = r_3, \quad \varepsilon^4_5 = -p_3, \quad \varepsilon^5_3 = -q_3, \quad \varepsilon^5_4 = p_3$$

where κ^s_α are the projections of $\mathbf{K}_\alpha = (\Delta_{\alpha1}/R)\mathbf{e}_1 + (\Delta_{\alpha2}/R)\mathbf{e}_2$ ($\alpha = 1, 2$) onto the $\mathbf{e}_1$ and $\mathbf{e}_2$ axes, γ^a_{bc} are the triple-index symbols from [1] and $d = c_{11}c_{22} - c_{12}c_{21} = \pm R, p_k, q_k, r_k$ are the coefficients in the formulae

$$\sigma_1 = p_1U^3 + p_2U^4 + p_3, \quad \tau_1 = q_1U^3 + q_2U^4 + q_3, \quad n_1 = r_1U^3 + r_2U^4 + r_3$$

Although the expressions obtained here for the quasi velocities are more complex than those of Lar'ye [3, p. 35], the symbols Γ^s_{tq}, ε^s_q ($s, t, q = 1, \ldots, m = 5$), as before, are defined by formulae (1.8.2) and (1.8.5) from [3], where the summation is carried out from 1 to m and, principally, all the symbols Γ^s_{tq}, ε^s_q in which one of indices exceeds $m = 5$ are equal to zero, and hence the equations of motion split into a group of Euler–Lagrange equations for the quasi velocities U^3, U^4 and U^5 and a group of Lagrange equations for the coordinates $\alpha^1, \ldots, \alpha^n$

$$\frac{d}{dt}\frac{\partial\Theta'}{\partial U^k} + \sum_{r=1}^{5}\sum_{t=3}^{5}\Gamma^r_{tk}\frac{\partial T'}{\partial U^r}U^t + \sum_{r=1}^{5}\frac{\partial T'}{\partial U^r}\varepsilon^r_k - \frac{\partial\Theta'}{\partial V^k} = P'_k \tag{2.3}$$

$$\frac{d}{dt}\frac{\partial\Theta'}{\partial\dot\alpha^s} - \frac{\partial\Theta'}{\partial\alpha^s} = Q_s; \quad k = 3, 4, 5; \quad s = 1, \ldots, n \tag{2.4}$$

The equations of motion of the supporting body (2.3) have the form ($Q_{rk} = \mathbf{Q}_r \cdot \mathbf{e}_k$)

$$\frac{d}{dt}\frac{\partial\Theta'}{\partial\sigma} + (\tau - \tau_1)\frac{\partial\Theta'}{\partial n} - (n - n_1)\frac{\partial\Theta'}{\partial\tau} +$$
$$+ M[(\xi - \xi_C)\tau - (\eta - \eta_C)\sigma]\sqrt{a_{22}}\dot q^2 + (Mj_3 + Q_{r3})\sqrt{a_{22}}\dot q^2 -$$
$$-(Q_{r2}j_3 - Q_{r3}j_2) + M[(n\xi - \sigma\varepsilon)j_3 - (\sigma\eta - \tau\xi)j_2] -$$
$$-M[(n\xi_C - \sigma\varepsilon_C)j_3 - (\sigma\eta_C - \tau\xi_C)j_2] = P'_3$$

$$\frac{d}{dt}\frac{\partial\Theta'}{\partial\tau}+(n-n_1)\frac{\partial\Theta'}{\partial\sigma}-(\sigma-\sigma_1)\frac{\partial\Theta'}{\partial n}- \tag{2.5}$$

$$-M[(\xi-\xi_C)\tau-(\eta-\eta_C)\sigma]\sqrt{a_{11}}\dot{q}^1-(Mj_3+Q_{r3})\sqrt{a_{11}}\dot{q}^1-$$

$$-(Q_{r3}j_1-Q_{r1}j_3)+M[(\sigma\eta-\tau\xi)j_1-(\tau\varepsilon-n\eta)j_3]-$$

$$-M[(\sigma\eta_C-\tau\xi_C)j_1-(\tau\varepsilon_C-n\eta_C)j_3] = P'_4$$

$$\frac{d}{dt}\frac{\partial\Theta'}{\partial n}+(\sigma-\sigma_1)\frac{\partial\Theta'}{\partial\tau}-(\tau-\tau_1)\frac{\partial\Theta'}{\partial\sigma}+M(\varepsilon-\varepsilon_C)(\sqrt{a_{11}}\dot{q}^1\sigma+\sqrt{a_{22}}\dot{q}^2\tau)-$$

$$-M[(\xi-\xi_C)\sqrt{a_{11}}\dot{q}^1+(\eta-\eta_C)\sqrt{a_{22}}\dot{q}^2]n-(Mj_1+Q_{r1})\sqrt{a_{22}}\dot{q}^2+$$

$$+(Mj_2+Q_{r2})\sqrt{a_{11}}\dot{q}^1-(Q_{r1}j_2-Q_{r2}j_1)+$$

$$+M[(\tau\varepsilon-n\eta)j_2-(n\xi-\sigma\varepsilon)j_1]-M[(\tau\varepsilon_C-n\eta_C)j_2-(n\xi_C-\sigma\varepsilon_C)j_1] = P'_5$$

The virtual displacement of any point M_1 of the system will be [3, p. 428]

$$\delta\mathbf{r}_i^a = \delta\mathbf{r}_0^a+\delta\mathbf{r}_i = \delta\mathbf{r}_0^a+\sum_{s=1}^{n}\frac{\partial r_i}{\partial\alpha^s}\delta\alpha^s+\boldsymbol{\theta}\times\mathbf{r}_i$$

The elementary work of all the active forces, applied both to the supporting and the supported bodies, on the virtual displacement of points of the system is

$$\delta'W = \sum_{i=1}^{N}\mathbf{F}_i\cdot\delta\mathbf{r}_i^a = \sum_{i=1}^{N}\mathbf{F}_i(\delta\mathbf{r}_0^a+\boldsymbol{\theta}\times\mathbf{r}_i)+\sum_{i=1}^{N}\mathbf{F}_i\left(\sum_{s=1}^{n}\frac{\partial\mathbf{r}_i}{\partial\alpha^s}\delta\alpha^s\right) =$$

$$= \mathbf{F}\cdot\delta\mathbf{r}_0^a+\mathbf{m}^O\cdot\boldsymbol{\theta}+\sum_{s=1}^{n}\left(\sum_{i=1}^{N}\mathbf{F}_i\frac{\partial\mathbf{r}_i}{\partial\alpha^s}\right)\delta\alpha^s$$

where $\mathbf{F}$ is the principal vector and $\mathbf{m}^0$ is the principal moment of all the active forces about the pole O. In the general case, each force $\mathbf{F}_i$ is the sum of a potential force $\mathbf{F}_{ip}$ and a non-potential force $\mathbf{F}_{in}$.

Taking the equation of non-holonomic constraint into account, we obtain from (1.6)

$$\delta\mathbf{r}_0^a+\boldsymbol{\theta}\times\boldsymbol{\rho}-\delta\mathbf{r}_{O_c}^a-\boldsymbol{\theta}_c\times\boldsymbol{\rho}^c = 0 \tag{2.6}$$

Since $\boldsymbol{\theta}_c = 0$, $\delta\mathbf{r}_{O_c}^a = 0$, we have $\delta\mathbf{r}_0^a = \boldsymbol{\rho}\times\boldsymbol{\theta}$, whence

$$\delta'W = \mathbf{F}\cdot(\boldsymbol{\rho}\times\boldsymbol{\theta})+\mathbf{m}^O\cdot\boldsymbol{\theta}+\sum_{s=1}^{n}\left(\sum_{i=1}^{N}\mathbf{F}_i\frac{\partial\mathbf{r}_i}{\partial\alpha^s}\right)\delta\alpha^s =$$

$$= (\mathbf{m}^O-\boldsymbol{\rho}\times\mathbf{F})\cdot(\delta V^3\mathbf{e}_1+\delta V^4\mathbf{e}_2+\delta V^5\mathbf{e}_3)+\sum_{s=1}^{n}\left(\sum_{i=1}^{N}\mathbf{F}_i\frac{\partial\mathbf{r}_i}{\partial\alpha^s}\right)\delta\alpha^s \tag{2.7}$$

On the other hand, $\delta'W$ is expressed in terms of generalized forces, referred to the quasicoordinates V^3, V^4 and V^5 and to the generalized coordinates α^k

$$\delta'W = \sum_{k=3}^{5}P'_k\delta V^k+\sum_{s=1}^{n}Q_s\delta\alpha^s \tag{2.8}$$

Comparing expressions (2.7) and (2.8) we find that the quantities P'_3, P'_4 and P'_5 are the projections onto the axis Mq^1q^1n of the principal moment of the active forces (applied both to the supporting and the supported bodies) about the contact point M

$$P'_{2+k} = \mathbf{m}^M \cdot \mathbf{e}_k (k = 1, 2, 3; \mathbf{m}^M = \mathbf{m}^O - \boldsymbol{\rho} \times \mathbf{F}), \quad Q_s = \sum_{i=1}^{N} \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial \alpha^s}$$

Carrying out the above calculations separately for the forces $\mathbf{F}_{ip}$ (and the forces $\mathbf{F}_{in}$), we obtain that the quantities $\partial U/\partial V^3$, $\partial U/\partial V^4$, $\partial U/\partial V^5$, which have the form (2.14) from [1, p. 809] (and the generalized forces P'_{3n}, P'_{4n} and P'_{5n}, produced by the non-potential forces $\mathbf{F}_{in}$), are respectively the projections onto the Mq^1q^1n axis of the principal moment of the active potential forces (the active non-potential forces) about the contact point M. Also $P'_k = \partial U/\partial V^k + P'_{kn}$ $(k = 3, 4, 5)$.

For a fixed surfaces S^c ($\mathbf{j} = 0$) the equations of motion of the supportion body (2.5) are identical with Eqs (2.12) and (2.13) from [1]. In vector form, these equations are

$$\begin{aligned} &\frac{d}{dt}[\boldsymbol{\Theta}^O \cdot \boldsymbol{\omega} + M\rho^2\boldsymbol{\omega} - M(\boldsymbol{\rho} \cdot \boldsymbol{\omega})\boldsymbol{\rho} - 2M(r_C \cdot \boldsymbol{\rho})\boldsymbol{\omega} + M(\boldsymbol{\omega} \cdot \boldsymbol{\rho})\mathbf{r}_C + \\ &+ M(\boldsymbol{\omega} \cdot \mathbf{r}_C)\boldsymbol{\rho} + \mathbf{Q}_r \times \boldsymbol{\rho} + \mathbf{K}_r^O - M(\boldsymbol{\rho} \times \mathbf{j}) + M(\mathbf{r}_C \times \mathbf{j})] + \\ &+ \overset{*}{\boldsymbol{\rho}} \times M[(\boldsymbol{\rho} - \mathbf{r}_C) \times \boldsymbol{\omega} + \overset{*}{\mathbf{r}}_C + \mathbf{j}] - (M\overset{*}{\mathbf{r}}_C \times \mathbf{j}) + M[\boldsymbol{\omega} \times (\boldsymbol{\rho} - \mathbf{r}_C)] \times \mathbf{j} = \mathbf{m}^M \end{aligned} \tag{2.9}$$

Thus, if the motion of the supported bodies ith respect to the supporting body is specified or there are generally no relative motions, we have obtained a closed system of eight differential equations (1.8), (1.10) and (2.5) for determining the generalized coordinates q^1, q^2, ϑ, q_c^1, q_c^2 and the quantities σ, τ, n as functions of time. In general, this system is not closed, and for a complete solution of the problem – to determine the generalized coordinates $q^1, q^2, \vartheta, q_c^1, q_c^2, \alpha^1, \ldots, \alpha^n$ and the quantities σ, τ, n as functions of time, we must add the equations of motion of the supported bodies (2.4) to Eqs (1.8), (1.10) and (2.5). Omitting the derivation, which can be found in [3, p. 433], we will immediately write Eqs (2.4) in the converted form

$$\begin{aligned} &\varepsilon_s(T_r) = Q_s - M[(\mathbf{j} + \boldsymbol{\rho} \times \boldsymbol{\omega})^* + \boldsymbol{\omega} \times (\mathbf{j} + \boldsymbol{\rho} \times \boldsymbol{\omega})] \cdot \frac{\partial \mathbf{r}_C}{\partial \alpha^s} + \\ &+ \frac{1}{2}\boldsymbol{\omega} \cdot \frac{\partial \boldsymbol{\Theta}^O}{\partial \alpha^s} \cdot \boldsymbol{\omega} - \dot{\boldsymbol{\omega}} \cdot \frac{\partial \mathbf{K}_r^O}{\partial \dot{\alpha}^s} - \boldsymbol{\omega} \cdot \varepsilon_s^*(\mathbf{K}_r^O), \quad s = 1, \ldots, n \end{aligned} \tag{2.10}$$

If the motion of the surface S^c is specified, and the set of supported point masses is a rigid body, the equations of motion of supported body will be ($s = 1, 2, 3$; $k = 4, 5, 6$; [1, p. 811] and [3, pp. 454–458])

$$M_r \mathbf{W}_{C_r} \cdot \frac{\partial \mathbf{r}_{C_r}}{\partial \alpha^s} = Q_s \left(\mathbf{v}_0 = \mathbf{j} + \boldsymbol{\rho} \times \boldsymbol{\omega}, \boldsymbol{\omega}_r = \sum_{k=4}^{6} \mathbf{e}'_k \dot{\alpha}^k, \boldsymbol{\Theta}^O = \boldsymbol{\Theta}_0^O + \boldsymbol{\Theta}_r^O \right)$$

$$\mathbf{e}'_k \cdot \left[\boldsymbol{\Theta}_r^{C_r} \cdot \overset{*}{\boldsymbol{\omega}}_r + \boldsymbol{\omega}_r \times \boldsymbol{\Theta}_r^{C_r} \cdot \boldsymbol{\omega}_r + \boldsymbol{\Theta}_r^{C_r} \cdot \overset{*}{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{\Theta}_r^{C_r} \cdot \boldsymbol{\omega} + 2\boldsymbol{\omega}_r \times \left(\boldsymbol{\Theta}_r^{C_r} - \frac{1}{2}\mathbf{E}\vartheta_r^{C_r} \right) \cdot \boldsymbol{\omega} \right] = Q_k$$

$$\boldsymbol{\Theta}_r^O = \boldsymbol{\Theta}_r^{C_r} + M_r(\mathbf{E}\mathbf{r}_{C_r} \cdot \mathbf{r}_{C_r} - \mathbf{r}_{C_r}\mathbf{r}_{C_r})$$

Θ_0^O is the inertia tensor of the supporting body at the point O, $\boldsymbol{\Theta}_r^O$ is the inertia tensor of the supported body at the point O, $\boldsymbol{\Theta}_r^{C_r}$ is the inertia tensor of the supported body at its centre of inertia, $\vartheta_r^{C_r}$ is the is the sum of the diagonal components of the tensor $\boldsymbol{\Theta}_0^{C_r}$, $\mathbf{W}_{C_r}$ is the absolute acceleration of the centre of inertia C_r of the supported body, and $\boldsymbol{\omega}_r$ is the vector of the angular velocity of the rectangular axes of the coordinates $C_r x^r y^r z^r$, connected with the supported body, about the axes $Oxyz$.

Proceeding as in [3, p. 159], we obtain the angular momentum $\mathbf{K}^O$ of the system (the supporting body plus the supported points) about the moving pole O in absolute motion (in the case considered by the Lur'yv it was a fixed pole)

$$\mathbf{K}^O = M\mathbf{r}_C \times \mathbf{v}_0 + \boldsymbol{\Theta}^O \cdot \boldsymbol{\omega} + \mathbf{K}_r^O$$

If the pole O is the centre of inertia C, then $\mathbf{K}^O = \boldsymbol{\Theta}^O \cdot \boldsymbol{\omega} + \mathbf{K}_r^O$.

We will further determine the angular momentum about the contact point M of the system, consisting of the supporting body and the supported points, in absolute motion about fixed axes $O_a x^a y^a z^a$

$$\mathbf{K}^M = \mathbf{K}^O + \mathbf{Q} \times \overline{\mathbf{OM}} = M\mathbf{r}_C \times \mathbf{v}_0 + \mathbf{\Theta}^O \cdot \boldsymbol{\omega} + \mathbf{K}_r^O +$$
$$+ [M(\mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_C) + \mathbf{Q}_r] \times \boldsymbol{\rho},$$
$$\mathbf{Q}_r = M\overset{*}{\mathbf{r}}_C$$

Hence, substituting $\mathbf{v}_0 = \mathbf{j} - \boldsymbol{\omega} \times \boldsymbol{\rho}$, we obtain $\mathbf{K}^M = \mathbf{m}'$, i.e. $\partial\Theta'/\partial\sigma$, $(\partial\Theta'/\partial\tau)$, $\partial\Theta'/\partial n$ are the projections of the angular momentum $\mathbf{K}^M$ on to the axes to the axes $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

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